3. Application of Derivatives

- For the curve $y = f(x)$, the slope of tangent at the point (x_0, y_0) is given by $\left.\frac{dy}{dx}\right]_{(x_0, y_0)}$ or $f'(x_0)$.
- For the curve $y = f(x)$, the slope of normal at the point (x_0, y_0) is given by $\frac{-1}{\left.\frac{dy}{dx}\right]_{(x_0, y_0)}}$ or $\frac{-1}{f'(x_0)}$.
- The equation of tangent to the curve $y = f(x)$ at the point (x_0, y_0) is given by, $y - y_0 = f'(x_0) \times (x - x_0)$
- If $f'(x_0)$ does not exist, then the tangent to the curve $y = f(x)$ at the point (x_0, y_0) is parallel to the y-axis and its equation is given by $x = x_0$.
- The equation of normal to the curve $y = f(x)$ at the point (x_0, y_0) is given by, $y - y_0 = \frac{-1}{f'(x_0)}\left(x - x_0\right)$
- If $f'(x_0)$ does not exist, then the normal to the curve $y = f(x)$ at the point (x_0, y_0) is parallel to the x-axis and its equation is given by $y = y_0$.
- If $f'(x_0) = 0$, then the respective equations of the tangent and normal to the curve $y = f(x)$ at the point (x_0, y_0) are $y = y_0$ and $x = x_0$.

- For a quantity y varying with another quantity x, satisfying the rule $y = f(x)$, the rate of change of y with respect to x is given by $\frac{dy}{dx}$ or $f'(x)$

  The rate of change of y with respect to x at the point $x = x_0$ is given by $\left.\frac{dy}{dx}\right]_{x = x_0}$ or $f'(x_0)$.

- If the variables x and y are expressed in form of $x = f(t)$ and $y = g(t)$, then the rate of change of y with respect to x is given by $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$ provided $f'(t) \neq 0$

- Let $y = f(x)$ and let Δx be a small increment in x and Δy be the increment in y corresponding to the increment in x i.e., $\Delta y = f(x + \Delta x) - f(x)$

  Then, $dy = f'\left(x\right)dx$ or $dy = \left(\frac{dy}{dx}\right)\Delta x$ is a good approximation of Δy, when $dx = \Delta x$ is relatively small and we denote it by $dy \approx \Delta y$.

- **Rolle's Theorem:**
  If f: $[a, b] \to \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, where a and b are some real numbers, then there exists some $c \in (a, b)$ such that $f'(c) = 0$

  Example: Verify Rolle's Theorem for the function:
  $f(x) = 2x^2 - 17x + 30$ in the interval $\left[\frac{5}{2}, 6\right]$.

  Solution:
  $f(x) = 2x^2 - 17x + 30$
  $\therefore f'(x) = 4x - 17$
  The function $f(x)$ being a polynomial, is continuous on $\left[\frac{5}{2}, 6\right]$ and is differentiable on $\left(\frac{5}{2}, 6\right)$.

Also, $f\left(\frac{5}{2}\right)=2\left(\frac{5}{2}\right)^2-17\left(\frac{5}{2}\right)+30=0$

And, $f(6)=2(6)^2-17\times 6+30=0$

$\therefore f\left(\frac{5}{2}\right)=f(6)$

Therefore, we can apply Rolle's Theorem for $f(x)$.

According to this theorem, there exists $c\in\left(\frac{5}{2},6\right)$ such that $f'(c)=0$

We have $f'(x)=4x-17$

$\therefore f'(c)=0$

$\Rightarrow 4c-17=0$

$\Rightarrow c=\frac{17}{4}\in\left(\frac{5}{2},6\right)$

Therefore, Rolle's Theorem is verified.

- **Mean value theorem:**

If f: $[a, b] \rightarrow \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

Example: Verify Mean Value Theorem for the function:

$f(x) = 2x^2 - 17x + 30$ in the interval $\left[\frac{5}{2},6\right]$.

Solution:

$f(x) = 2x^2 - 17x + 30$

$\therefore f'(x)=4x-17$

The function $f(x)$ being a polynomial, is continuous on $\left[\frac{5}{2},6\right]$ and is differentiable on $\left(\frac{5}{2},6\right)$.

Also, $f\left(\frac{5}{2}\right)=2\left(\frac{5}{2}\right)^2-17\left(\frac{5}{2}\right)+30=0$

And, $f(6)=2(6)^2-17\times 6+30=0$

$\therefore f\left(\frac{5}{2}\right)=f(6)$

Now, $\frac{f(6)-f\left(\frac{5}{2}\right)}{6-\frac{5}{2}}=0$

According to Mean Value Theorem (MVT), there exists $c\in\left(\frac{5}{2},6\right)$ such that $f'(c)=0$

$\therefore 4c-17=0$

$\Rightarrow c=\frac{17}{4}\in\left(\frac{5}{2},6\right)$

Therefore, M.V.T is verified.

a. A function f: $(a, b) \rightarrow \mathbf{R}$ is said to be

- increasing on (a, b), if $x_1 < x_2$ in (a, b)
- decreasing on (a, b), if $x_1 < x_2$ in (a, b)

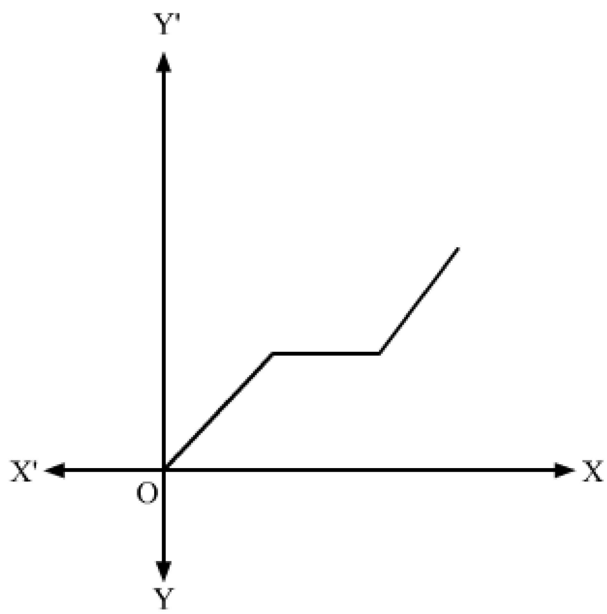
OR

If a function f is continuous on $[a, b]$ and differentiable on (a, b), then

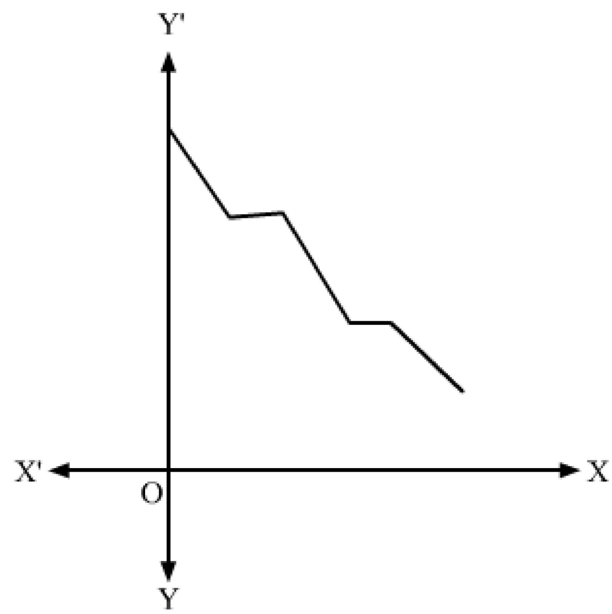
- f is increasing in $[a, b]$, if $f'(x) > 0$ for each $x \in (a, b)$
- f is decreasing in $[a, b]$, if $f'(x) < 0$ for each $x \in (a, b)$
- f is constant function in $[a, b]$, if $f'(x) = 0$ for each $x \in (a, b)$

a. A function $f: (a, b) \to \mathbf{R}$ is said to be
   - strictly increasing on (a, b), if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) < f(x_2)\ \forall\ x_1, x_2 \in (a, b)$
   - strictly decreasing on (a, b), if $x_1 < x_2$ in $(a, b) \Rightarrow f(x_1) > f(x_2)\ \forall\ x_1, x_2 \in (a, b)$

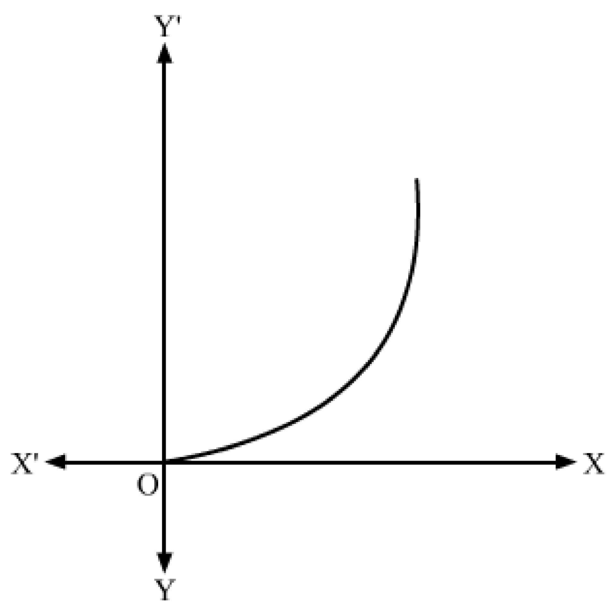
   a. The graphs of various types of functions can be shown as follows:



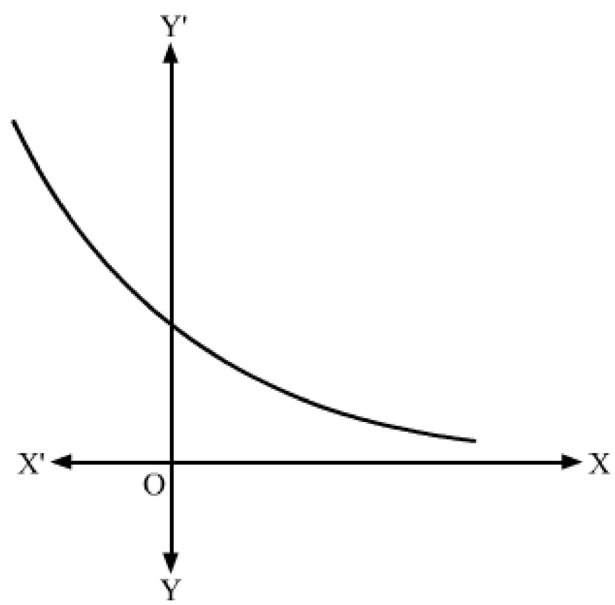
Increasing Function



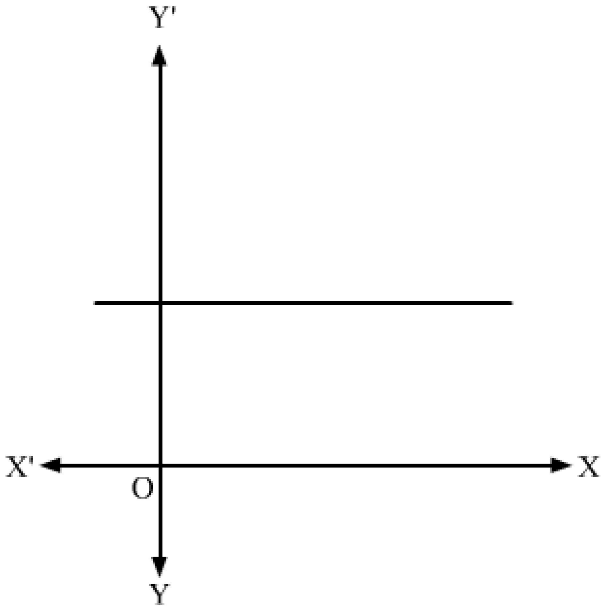
Decreasing Function



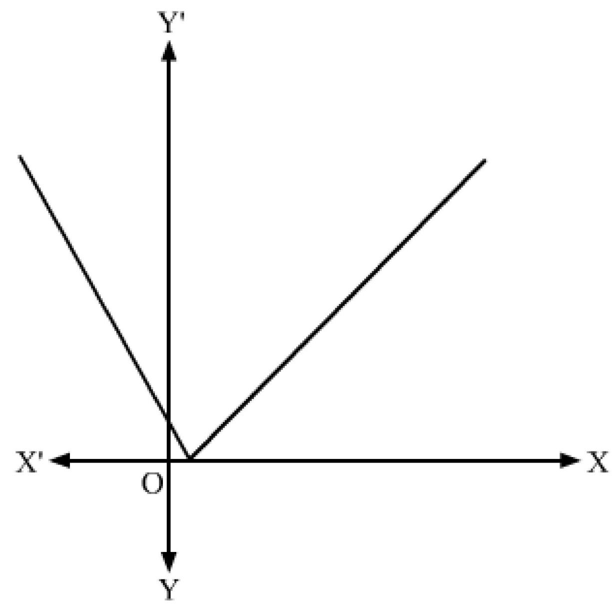
Strictly Increasing Function



Strictly Decreasing Function



Constant Function



Neither increasing or dreasing function

Example 1: Find the intervals in which the function f given by $f(x)=\sqrt{3}\sin x-\cos x, x\in[0,\ 2\pi]$ is strictly increasing or decreasing.

Solution:

$$f(x)=\sqrt{3}\sin x-\cos x$$
$$\therefore f'(x)=\sqrt{3}\cos x+\sin x$$
$$f'(x)=0 \text{ gives } \tan x=-\sqrt{3}$$

The points $x=\frac{2\pi}{3}, \frac{5\pi}{3}$ $\frac{2\pi}{3}$ and $x=\frac{5\pi}{3}$ divide the interval [0, 2π] into three disjoint intervals,

$$\left[0,\ \frac{2\pi}{3}\right),\ \left(\frac{2\pi}{3},\ \frac{5\pi}{3}\right),\ \left(\frac{5\pi}{3},\ 2\pi\right].$$

Now, $f'(x)>0$, if $x\in\left[0,\ \frac{2\pi}{3}\right)\cup\left(\frac{5\pi}{3},\ 2\pi\right]$

f is strictly increasing in the intervals $\left[0,\ \frac{2\pi}{3}\right)$ and $\left(\frac{5\pi}{3},\ 2\pi\right]$.

Also, $f'(x)<0$, if $x\in\left(\frac{2\pi}{3},\ \frac{5\pi}{3}\right)$

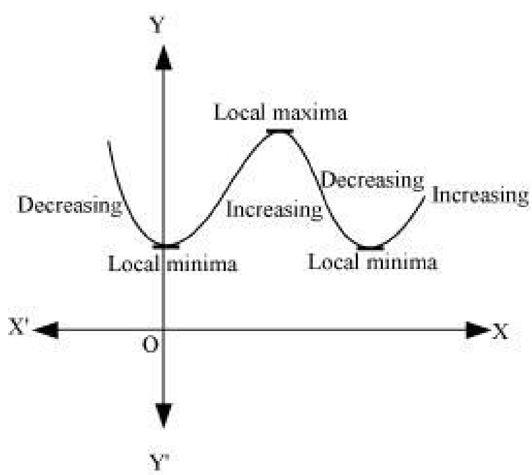
f is strictly decreasing in the interval $\left(\frac{2\pi}{3},\ \frac{5\pi}{3}\right)$.

- **Maxima and Minima**: Let a function f be defined on an interval I. Then, f is said to have

- maximum value in I, if there exists $c \in$ I such that $f(c) > f(x)$, $\forall\ x \in$ I [In this case, c is called the point of maxima]
- minimum value in I, if there exists $c \in$ I such that $f(c) < f(x)$, $\forall\ x \in$ I [In this case, c is called the point of minima]
- an extreme value in I, if there exists $c \in$ I such that c is either point of maxima or point of minima [In this case, c is called an extreme point]

Note: Every continuous function on a closed interval has a maximum and a minimum value.

- **Local maxima and local minima:** Let f be a real-valued function and c be an interior point in the domain of f. Then, c is called a point of
  - local maxima, if there exists $h > 0$ such that $f(c) > f(x)$, $\forall\ x \in (c - h, c + h)$ [In this case, $f(c)$ is called the local maximum value of f]
  - local minima, if there exists $h > 0$ such that $f(c) < f(x)$, $\forall\ x \in (c - h, c + h)$ [In this case, $f(c)$ is called the local maximum value of f]



- A point c in the domain of a function f at which either $f'(c) = 0$ or f is not differentiable is called a critical point of f.

- **First derivative test:** Let f be a function defined on an open interval I. Let f be continuous at a critical point c in I. Then:
  - If $f'(x)$ changes sign from positive to negative as x increases through c, i.e. if $f'(x) > 0$ at every point sufficiently close to and to the left of c, and $f'(x) < 0$ at every point sufficiently close to and to the right of c, then c is a point of local maxima.
  - If $f'(x)$ changes sign from negative to positive as x increases through c, i.e. if $f'(x) < 0$ at every point sufficiently close to and to the left of c, and $f'(x) > 0$ at every point sufficiently close to and to the right of c, then c is a point of local minima.
  - If $f'(x)$ does not change sign as x increases through c, then c is neither a point of local maxima nor a point of local minima. Such a point c is called point of inflection.

- **Second derivative test:** Let f be a function defined on an open interval I and $c \in$ I. Let f be twice differentiable at c and $f'(c) = 0$ Then:
  - If $f'(c) < 0$, then c is a point of local maxima. In this situation, $f(c)$ is local maximum value of f.
  - If $f'(c) > 0$, then c is a point of local minima. In this situation, $f(c)$ is local minimum value of f.
  - If $f'(c) = 0$, then the test fails. In this situation, we follow first derivative test and find whether c is a point of maxima or minima or a point of inflection.

Example 1: Find all the points of local maxima or local minima of the function f given by $f(x) = x^3 - 12x^2 + 36x - 4$.

Solution:
We have,

$f(x) = x^3 - 12x^2 + 36x - 4$

$\therefore f'(x) = 3x^2 - 24x + 36 = 3\left(x^2 - 8x + 12\right)$

and $f''(x) = 3(2x - 8) = 6(x - 4)$

Now, $f'(x) = 0$ gives $x^2 - 8x + 12 = 0$

$\Rightarrow (x - 2)(x - 6) = 0$

$\Rightarrow x = 2$ or $x = 6$

However, $f''(2) = -12$ and $f''(6) = 12$

Therefore, the point of local maxima and local minima are at the points $x = 2$ and $x = 6$ respectively.
The local maximum value is $f(2) = 28$
The local minimum value is $f(6) = -4$